

# Polynomial symmetries of spherical Lissajous systems

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## Abstract

In a previous work, both the constants of motion of a classical system and the symmetries of the corresponding quantum version have been computed with the help of factorizations. As their expressions were not polynomial, in this paper the question of finding an equivalent set of polynomial constants of motion and symmetries is addressed. The general algebraic relations as well as the appropriate Hermitian relations will also be found.

## 1 Introduction

In a former paper [1], henceforth referred as Lissajous–1, we have characterized a type of symmetries for a particular classical system that we called ‘Lissajous system on the sphere’. The name ‘Lissajous’ for this system (and many others that share this feature) comes from the form of its bounded trajectories: they are closed and similar to the usual Lissajous curves, where the motion in the variables have a rational frequency. We have found what we will call a set of ‘fundamental constants of motion’ (sometimes they will be referred as classical symmetries) given by four functions  $\langle H, H_\phi, X^\pm \rangle$ , where  $H$  and  $H_\phi$  are trivial symmetries due the separation of the two variables, while  $X^\pm$  is a pair of complex nontrivial symmetries. In total there are three independent symmetries, but they are not polynomial since  $X^\pm$  include square roots of some functions. Such roots are well defined for any physical motion of the system.

In the same paper, we have considered the corresponding quantum Lissajous system on the sphere. We have computed another set of ‘fundamental symmetries’ that consist of four operators  $\langle \hat{H}, \hat{H}_\phi, \hat{X}^\pm \rangle$ , which are in correspondence with the above classical symmetries. Again, some square roots of operators are present, but such expressions are well defined in the physical space generated by eigenfunctions.

We have applied a general unifying method that was introduced in Ref. [2] to compute the fundamental symmetries of classical and quantum systems. This method is based on the well known factorizations of quantum mechanics [3, 4] and a classical version called ‘classical factorizations’ [5].

Now, in this paper we will show that the above mentioned fundamental symmetries determine in a straightforward way the polynomial symmetries of both classical and quantum systems. Therefore, in this way we can eliminate the somewhat ‘problematic’ square roots and state our results in the frame of polynomial superintegrable systems. Besides, we will easily get the algebraic structure that such polynomial symmetries will close, as well as their Hermitian properties.

Section 2 will be devoted to polynomial symmetries of the classical system, while the next one will implement the same approach to get the polynomial symmetries of the quantum system. We will end with some conclusions, in particular, we will also comment our results in the light of other previous contributions in the literature on the symmetries of this type of systems. We have included all the basic results of Lissajous–1, so that the present paper is self-contained and can be read independently.

## 2 The classical system

We will consider a classical system on the sphere (of unit radius) whose Hamiltonian is

$$H = p_\theta^2 + \frac{1}{\sin^2 \theta} p_\varphi^2 + \frac{k^2 \alpha^2}{\sin^2 \theta \cos^2 k\varphi}, \quad (1)$$

where  $0 < \theta < \pi$  and  $-\pi/(2k) < \varphi < \pi/(2k)$ . In spherical coordinates, the first two terms correspond to the kinetic part of the system, while the last one is for the potential. We will assume that  $k \geq 1/2$ , in order that this system be well defined on the sphere. After a change of canonical variables  $\phi = k\varphi$ ,  $p_\phi = p_\varphi/k$  we get an equivalent system given by the Hamiltonian

$$H = p_\theta^2 + \frac{k^2}{\sin^2 \theta} \left( p_\phi^2 + \frac{\alpha^2}{\cos^2 \phi} \right), \quad (2)$$

where now the range of the variable  $\phi$  is  $-\pi/2 < \phi < \pi/2$ .

### 2.1 Ladder and shift functions

This Hamiltonian is explicitly separated in the variables  $\theta$  and  $\phi$ , so that two constants of motion will be identified with two one-dimensional Hamiltonians,

$$H_\phi = p_\phi^2 + \frac{\alpha^2}{\cos^2 \phi} \quad (3)$$

and

$$H_\theta^M = p_\theta^2 + \frac{k^2 H_\phi}{\sin^2 \theta} = p_\theta^2 + \frac{M^2}{\sin^2 \theta}, \quad M = k^2 H_\phi. \quad (4)$$

In order to build nontrivial symmetries we need two types of functions related to the above two Hamiltonians.

The first pair of functions  $B^\pm(\phi, p_\phi)$  are called ladder functions for  $H_\phi$ . They are given by [5]

$$B^\pm = \mp i \cos \phi p_\phi + \sqrt{H_\phi} \sin \phi. \quad (5)$$

For a real motion where  $H_\phi > 0$ , they are complex conjugate functions. These ladder functions factorize the Hamiltonian in the form

$$H_\phi = B^+ B^- + \alpha^2. \quad (6)$$

The set  $\langle H_\phi, B^\pm \rangle$  satisfy the following PBs,

$$\{H_\phi, B^\pm\} = \mp 2i \sqrt{H_\phi} B^\pm, \quad \{B^-, B^+\} = -2i \sqrt{H_\phi}. \quad (7)$$

The second set of functions  $A^\pm(\theta, p_\theta, M)$  are called shift functions of  $H_\theta^M$ . They are also formed by two complex conjugate functions (for a real motion) given by

$$A^\pm = \mp i p_\theta + M \cot \theta. \quad (8)$$

In this case, they factorize the Hamiltonian  $H_\theta^M$  in the form

$$H_\theta^M = A^+ A^- + M^2. \quad (9)$$

The functions  $A^\pm$  and  $H_\theta^M$  obey the following PBs

$$\{H_\theta^M, A^\pm\} = \pm 2i \frac{M}{\sin^2 \theta} A^\pm, \quad \{A^-, A^+\} = 2i \frac{M}{\sin^2 \theta}. \quad (10)$$

## 2.2 The algebra of fundamental and polynomial symmetries

If the parameter  $k$  that appear in the initial Hamiltonian (2) is rational,  $k = m/n$ , where  $m, n \in \mathbb{N}$ , then a pair of symmetries  $X^\pm$ , such that  $\{H, X^\pm\} = 0$ , is built in terms of the above sets of ladder and shift functions [1],

$$X^\pm = (B^\pm)^n (A^\pm)^m. \quad (11)$$

Therefore, at this stage we have four ‘fundamental symmetries’  $\langle H, H_\phi, X^\pm \rangle$  whose constant values will be denoted by  $E, E_\phi, Q^\pm$ , respectively. In fact, as  $X^\pm$  are complex conjugate functions, we can write  $Q^\pm = q e^{\pm i \phi_0}$ , where  $q$  is for the modulus and  $\phi_0$  is for the phase. The last two constants of motion are not independent since, due to the factorizations (6) and (9), their product is

$$X^+ X^- = (E_\phi - \alpha^2)^n (E - k^2 E_\phi)^m, \quad E_\phi \geq \alpha^2, E \geq k^2 E_\phi. \quad (12)$$

The algebraic structure of the fundamental symmetry set  $\langle H, H_\phi, X^\pm \rangle$  can be obtained by computing its PBs in a direct way from (7) and (10):

$$\begin{cases} \{H_\phi, X^\pm\} = \mp 2i n \sqrt{H_\phi} X^\pm, & \{H, H_\phi\} = \{H, X^\pm\} = 0 \\ \{X^+, X^-\} = 2i \sqrt{H_\phi} [-km^2(H_\phi - \alpha^2) + n^2(H - k^2 H_\phi)] (H_\phi - \alpha^2)^{n-1} (H - k^2 H_\phi)^{m-1}. \end{cases} \quad (13)$$

As we know from Lissajous-1, the symmetries  $X^\pm$  are quite appropriate to describe the trajectories of the motion for this system. In particular, we have shown that they are a kind of Lissajous curves on the sphere. However, this class of symmetries are not polynomial, due to the square root that takes part in the functions  $A^\pm$  and  $B^\pm$ , so we will show below how they can supply us the relevant polynomial symmetries.

Let us compute explicitly the expression of  $X^\pm$  by expanding the powers in (11),

$$X^\pm = (\mp i \cos \phi p_\phi + \sqrt{H_\phi} \sin \phi)^n (\mp i p_\theta + k \sqrt{H_\phi} \cot \theta)^m. \quad (14)$$

Depending on the parity type of  $m + n$  we will write the result according to the following convention.

(a) For  $m + n$  even we use the notation,

$$X^\pm = \pm i \mathcal{O} \sqrt{H_\phi} + \mathcal{E} \quad (15)$$

(b) For  $m + n$  odd we write the result in the form,

$$X^\pm = \mathcal{O} \sqrt{H_\phi} \mp i \mathcal{E}. \quad (16)$$

The functions  $\mathcal{O}(\phi, \theta, p_\phi, p_\theta)$  and  $\mathcal{E}(\phi, \theta, p_\phi, p_\theta)$  are real polynomial functions of  $p_\phi, p_\theta$ ; the first one comes from odd and the second from the even degrees of  $\sqrt{H_\phi}$  in the expansion (14). In both cases the polynomials (in  $p_\phi, p_\theta$ ),  $\mathcal{O}$  and  $\mathcal{E}$  are of degree  $n + m - 1$  and  $n + m$ , respectively. These two polynomial functions are symmetries of the initial Hamiltonian,

$$\{H, \mathcal{O}\} = \{H, \mathcal{E}\} = 0. \quad (17)$$

In conclusion, we have obtained an equivalent set of symmetries that will be called ‘polynomial symmetries’,

$$\langle H, H_\phi, \mathcal{O}, \mathcal{E} \rangle. \quad (18)$$

Its algebraic structure can be computed by inserting the expressions (15) or (16) in (13). Finally, we arrive at the following non-vanishing PBs

$$\begin{cases} \{H_\phi, \mathcal{O}\} = -2n \mathcal{E}, & \{H_\phi, \mathcal{E}\} = 2n H_\phi \mathcal{O} \\ \{\mathcal{O}, \mathcal{E}\} = -n \mathcal{O}^2 + [-km^2(H_\phi - \alpha^2) + n^2(H - k^2 H_\phi)] (H_\phi - \alpha^2)^{n-1} (H - k^2 H_\phi)^{m-1}. \end{cases} \quad (19)$$

The dependence relation of the four polynomial generators is obtained from (12),

$$\mathcal{O}^2 H_\phi + \mathcal{E}^2 = (H_\phi - \alpha^2)^n (H - k^2 H_\phi)^m. \quad (20)$$

In other words, the polynomial symmetries close a polynomial algebra (with respect to the generators  $H, H_\phi, \mathcal{O}, \mathcal{E}$ ), where all the square roots have been eliminated in the final expressions. For the case  $k = 1$ , this polynomial algebra will be quadratic, for other cases, the degree will be  $m + n - 1$ .

### 2.3 Examples of classical polynomial symmetry algebras

We will give some examples to illustrate the general results of the previous subsections. In this way we can see how are the concrete expressions of polynomial algebras for some simple cases. We have checked the following formulas with the help of *Mathematica*.

- Case  $m = 1, n = 1, k = 1$

$$\mathcal{O} = -p_\phi \cot \theta \cos \phi - \sin \phi p_\theta \quad (21)$$

$$\mathcal{E} = -\cos \phi p_\theta p_\phi + \cot \theta (p_\phi^2 \sin \phi + \alpha^2 \sec \phi \tan \phi) \quad (22)$$

$$\begin{aligned} \{H_\phi, \mathcal{O}\} &= -2\mathcal{E}, & \{H_\phi, \mathcal{E}\} &= 2H_\phi \mathcal{O} \\ \{\mathcal{O}, \mathcal{E}\} &= -\mathcal{O}^2 - (H_\phi - \alpha^2) + (H - H_\phi) \end{aligned} \quad (23)$$

- Case  $m = 1, n = 2, k = 1/2$

$$\mathcal{O} = \frac{1}{2} [-p_\phi^2 \cos^2 \phi \cot \theta - 4p_\theta p_\phi \cos \phi \sin \phi + \cot \theta (p_\phi^2 \sin^2 \phi + \alpha^2 \tan^2 \phi)] \quad (24)$$

$$\mathcal{E} = -p_\theta p_\phi^2 \cos^2 \phi + p_\phi^3 \cos \phi \cot \theta \sin \phi + p_\theta p_\phi^2 \sin^2 \phi + \alpha^2 \tan \phi (p_\phi \cot \theta + p_\theta \tan \phi) \quad (25)$$

$$\begin{aligned} \{H_\phi, \mathcal{O}\} &= -4\mathcal{E}, & \{H_\phi, \mathcal{E}\} &= 4H_\phi \mathcal{O} \\ \{\mathcal{O}, \mathcal{E}\} &= -2\mathcal{O}^2 + \left[-\frac{1}{2}(H_\phi - \alpha^2) + 4(H - \frac{1}{4}H_\phi)\right] (H_\phi - \alpha^2) \end{aligned} \quad (26)$$

- Case  $m = 2, n = 1, k = 2$

$$\mathcal{O} = -4p_\theta p_\phi \cos \phi \cot \theta - (p_\theta^2 - 4p_\phi^2 \cot^2 \theta) \sin \phi + 4\alpha^2 \cot^2 \theta \sec \phi \tan \phi \quad (27)$$

$$\mathcal{E} = -p_\phi \cos \phi (p_\theta^2 - 4p_\phi^2 \cot^2 \theta) + 4 \cot \theta [p_\phi \alpha^2 \cot \theta \sec \phi + p_\theta p_\phi^2 \sin \phi + p_\theta \alpha^2 \sec \phi \tan \phi] \quad (28)$$

$$\begin{aligned} \{H_\phi, \mathcal{O}\} &= -2\mathcal{E}, & \{H_\phi, \mathcal{E}\} &= 2H_\phi \mathcal{O} \\ \{\mathcal{O}, \mathcal{E}\} &= -\mathcal{O}^2 + [-8(H_\phi - \alpha^2) + (H - 4H_\phi)] (H - 4H_\phi) \end{aligned} \quad (29)$$

- Case  $m = 3, n = 1, k = 3$

$$\begin{aligned} \mathcal{O} = & 9p_\phi \cos \phi \cot \theta \left( p_\theta^2 - 3p_\phi^2 \cot^2 \theta \right) - 27p_\phi \alpha^2 \cot^3 \theta \sec \phi + p_\theta^3 \sin \phi \\ & - 27p_\theta \cot^2 \theta \left( p_\phi^2 \sin \phi + \alpha^2 \sec \phi \tan \phi \right) \end{aligned} \quad (30)$$

$$\begin{aligned} \mathcal{E} = & p_\theta p_\phi \cos \phi \left( p_\theta^2 - 27p_\phi^2 \cot^2 \theta \right) - 9 \cot \theta [3p_\theta p_\phi \alpha^2 \cot \theta \sec \phi \\ & - \frac{3}{4} \left( p_\phi^2 + 2\alpha^2 + p_\phi^2 \cos 2\phi \right)^2 \cot^2 \theta \sec^3 \phi \tan \phi + p_\theta^2 \left( p_\phi^2 \sin \phi + \alpha^2 \sec \phi \tan \phi \right)] \end{aligned} \quad (31)$$

$$\begin{aligned} \{H_\phi, \mathcal{O}\} &= -\mathcal{E}, \quad \{H_\phi, \mathcal{E}\} = H_\phi \mathcal{O} \\ \{\mathcal{O}, \mathcal{E}\} &= -\mathcal{O}^2 + [-27(H_\phi - \alpha^2) + (H - 9H_\phi)] (H - 9H_\phi)^2 \end{aligned} \quad (32)$$

### 3 The quantum system

In quantum mechanics the corresponding system is described by the Hamiltonian operator in spherical coordinates (the units  $2m = 1$  and  $\hbar = 1$  have been chosen):

$$\hat{H} = -\partial_\theta^2 - \cot \theta \partial_\theta - \frac{1}{\sin^2 \theta} \partial_\varphi^2 + \frac{k^2 \alpha^2}{\sin^2 \theta \cos^2 k \varphi} \quad (33)$$

where  $0 < \theta < \pi$  and  $0 < \varphi < \pi/2k$ . The first three terms describe the kinetic energy, they are obtained from the Laplacian in spherical coordinates; the last term is for the potential. After the change of variable  $\phi = k\varphi$ , we get a new equivalent Hamiltonian that we will use hereafter,

$$\hat{H} = -\partial_\theta^2 - \cot \theta \partial_\theta + \frac{k^2}{\sin^2 \theta} \left( -\partial_\phi^2 + \frac{\alpha^2}{\cos^2 \phi} \right). \quad (34)$$

The corresponding eigenvalue equation

$$\hat{H} \Psi(\theta, \phi) = E \Psi(\theta, \phi), \quad (35)$$

is a partial differential equation separated in the variables  $\theta$  and  $\phi$ . Then, we will look for separable solutions,  $\Psi(\theta, \phi) = \Theta_E^M(\theta) \Phi_\epsilon(\phi)$  characterized by

$$\hat{H}_\theta^M \Theta_E^M(\theta) = E_\theta \Theta_E^M(\theta), \quad \hat{H}_\phi \Phi_\epsilon(\phi) = E_\phi \Phi_\epsilon(\phi), \quad E_\theta := E, \quad E_\phi := \epsilon^2, \quad (36)$$

where

$$\hat{H}_\phi = -\partial_\phi^2 + \frac{\alpha^2}{\cos^2 \phi} \quad (37)$$

$$\hat{H}_\theta^M = -\partial_\theta^2 - \cot \theta \partial_\theta + \frac{1}{\sin^2 \theta} M^2, \quad M^2 := k^2 \epsilon^2. \quad (38)$$

The operators  $\hat{H}_\phi$  and  $\hat{H}_\theta^M$  can be considered as one-dimensional component Hamiltonians of the total Hamiltonian (34).

### 3.1 Ladder and shift operators

Since the Hamiltonian is separable, we already have a symmetry  $\hat{H}_\phi$  associated to this separation. As in the classical case, in order to get a nontrivial symmetry we need the ladder and shift operators of the one-dimensional Hamiltonians.

The ladder operators for  $\hat{H}_\phi$  are given by [6]

$$\hat{B}_\epsilon^+ = -\cos \phi \partial_\phi + \epsilon \sin \phi, \quad \hat{B}_\epsilon^- = \cos \phi \partial_\phi + (\epsilon + 1) \sin \phi. \quad (39)$$

They satisfy the following factorization relation

$$\hat{B}_\epsilon^- \hat{B}_\epsilon^+ = \hat{B}_{\epsilon+1}^+ \hat{B}_{\epsilon+1}^- = \epsilon(\epsilon + 1) - \alpha^2 \quad (40)$$

and their action on eigenfunctions is

$$\hat{B}_\epsilon^+ \Phi_\epsilon := \hat{B}_\epsilon^+ \Phi_\epsilon \propto \Phi_{\epsilon+1}, \quad \hat{B}_\epsilon^- \Phi_{\epsilon+1} := \hat{B}_\epsilon^- \Phi_{\epsilon+1} \propto \Phi_\epsilon. \quad (41)$$

Therefore, as they change the energy eigenvalues of eigenfunctions of the same Hamiltonian, they are called pure-ladder operators. It is convenient to use the free-index notation for operators, but at the same time, one must be careful with the rules (41) to act on eigenfunctions.

The shift operators for the Hamiltonian  $\hat{H}_\theta^M$  (38) are obtained by the standard factorization method [6]

$$\hat{H}_\theta^M = \hat{A}_M^+ \hat{A}_M^- + \lambda_M = \hat{A}_{M+1}^- \hat{A}_{M+1}^+ + \lambda_{M+1}, \quad \lambda_M = M(M - 1) \quad (42)$$

where

$$\hat{A}_M^+ = -\partial_\theta + (M - 1) \cot \theta, \quad \hat{A}_M^- = \partial_\theta + M \cot \theta. \quad (43)$$

The operators  $\hat{A}_M^\pm$  implement the following intertwining rules in the hierarchy (42) of Hamiltonians  $\{\hat{H}_\theta^M\}$ ,

$$\hat{A}_M^- \hat{H}_\theta^M = \hat{H}_\theta^{M-1} \hat{A}_M^-, \quad \hat{A}_M^+ \hat{H}_\theta^{M-1} = \hat{H}_\theta^M \hat{A}_M^+. \quad (44)$$

These rules can be written in a shorter notation by eliminating the subindex in  $\hat{A}^\pm$ , but taking care of their action. The operators  $\hat{A}^\pm$  keep the energy  $E_\theta$ , but change the parameter  $M$ , so they are called pure-shift operators.

### 3.2 Sets of fundamental and polynomial symmetries

As in the classical case if  $k = m/n$ ,  $m, n \in \mathbb{N}$ , then a pair of symmetry operators  $\hat{X}^\pm$  is given by

$$\hat{X}^\pm = (\hat{A}^\pm)^m (\hat{B}^\pm)^n. \quad (45)$$

The set of four symmetries  $\langle \hat{H}, \hat{H}_\phi, \hat{X}^\pm \rangle$  is called fundamental set. They are not independent; from the factorization properties (40) and (42) it can be seen that the products  $\hat{X}^+ \hat{X}^-$  and  $\hat{X}^- \hat{X}^+$

are functions of the diagonal operators  $\hat{H}, \hat{H}_\phi$ . We can explicitly compute them,

$$\begin{aligned}\hat{X}^+ \hat{X}^- &= \prod_{r=1}^n \left[ (\sqrt{\hat{H}_\phi - r})(\sqrt{\hat{H}_\phi - r + 1}) - \alpha^2 \right] \prod_{p=1}^m \left[ \hat{H} - (k\sqrt{\hat{H}_\phi - p})(k\sqrt{\hat{H}_\phi - p + 1}) \right] \\ &= P_1(\hat{H}, \hat{H}_\phi) - P_2(\hat{H}, \hat{H}_\phi) \sqrt{\hat{H}_\phi}\end{aligned}\quad (46)$$

$$\begin{aligned}\hat{X}^- \hat{X}^+ &= \prod_{r=1}^n \left[ (\sqrt{\hat{H}_\phi + r})(\sqrt{\hat{H}_\phi + r - 1}) - \alpha^2 \right] \prod_{p=1}^m \left[ \hat{H} - (k\sqrt{\hat{H}_\phi + p})(k\sqrt{\hat{H}_\phi + p - 1}) \right] \\ &= P_1(\hat{H}, \hat{H}_\phi) + P_2(\hat{H}, \hat{H}_\phi) \sqrt{\hat{H}_\phi},\end{aligned}\quad (47)$$

where the rhs. expressions are assumed to act on an eigenfunction of  $\hat{H}$  and  $\hat{H}_\phi$  characterized by the eigenvalues  $E$  and  $\sqrt{\hat{H}_\phi} = \epsilon$ . The functions  $P_1(\hat{H}, \hat{H}_\phi)$  and  $P_2(\hat{H}, \hat{H}_\phi)$  are polynomial in  $\hat{H}$  and  $\hat{H}_\phi$ . The quantum expressions (46) and (47) can be compared to the corresponding classical expression (12).

The symmetry operators  $\hat{X}^\pm$  act on the simultaneous eigenfunctions of  $\hat{H}$  and  $\hat{H}_\phi$ ; its action will give another common eigenfunction of both  $\hat{H}$  and  $\hat{H}_\phi$ . Therefore, they explain the degeneracy of each energy level and hence, the degeneracy of the spectrum of  $\hat{H}$ . We can easily compute the commutation relations of the fundamental symmetry set,

$$[\hat{H}_\phi, \hat{X}^\pm] = \hat{X}^\pm (\pm 2n \sqrt{\hat{H}_\phi} + n^2), \quad [\hat{X}^+, \hat{X}^-] = -2P_2(\hat{H}, \hat{H}_\phi) \sqrt{\hat{H}_\phi}. \quad (48)$$

The last commutator can be obtained simply by taking the difference of (46) and (47). If we add these two expressions we will get the constrain relation

$$\hat{X}^+ \hat{X}^- + \hat{X}^- \hat{X}^+ = 2P_1(\hat{H}, \hat{H}_\phi). \quad (49)$$

One may be tempted to discard the operators  $\hat{X}^\pm$  as ‘true’ symmetries since they depend on a formal square root operator  $\sqrt{\hat{H}_\phi}$ . However,  $\hat{X}^\pm$  are well defined operators in the space of eigenfunctions of  $\hat{H}$  (in fact, the operators  $\hat{X}^\pm$  are defined even in the space of formal eigenfunctions, physical or not, of the differential operator (34)). We can use them in order to get polynomial symmetries that exclude square roots. To show this, let us expand expression (45); here we will distinguish two cases.

(a) If  $m + n$  is even, the result of the expansion will be denoted by

$$\hat{X}^\pm = \pm \hat{\mathcal{O}} \sqrt{\hat{H}_\phi} + \hat{\mathcal{E}} \quad (50)$$

In this expression  $\hat{\mathcal{O}}$  and  $\hat{\mathcal{E}}$  are polynomial differential operators in  $\partial_\phi, \partial_\theta$ . Their degrees with respect to these partial derivatives are  $m + n - 1$ , and  $m + n$ , respectively.



(b) If  $m + n$  is odd, then the resulting expansion will take the form

$$\hat{X}^\pm = \hat{\mathcal{O}}\sqrt{\hat{H}_\phi} \pm \hat{\mathcal{E}}. \quad (51)$$

In this case the polynomial  $\hat{\mathcal{O}}$  and  $\hat{\mathcal{E}}$  operators have the same degrees  $m + n - 1$  and  $m + n$ .

The explicit expressions of the polynomial differential operators  $\hat{\mathcal{O}}$  and  $\hat{\mathcal{E}}$  can be obtained in closed form, but this will not be needed in what follows. These operators are symmetries of the initial Hamiltonian (34),

$$[\hat{H}, \hat{\mathcal{O}}] = [\hat{H}, \hat{\mathcal{E}}] = 0. \quad (52)$$

Hence, we have arrived at a set of polynomial symmetries

$$\langle \hat{H}, \hat{H}_\phi, \hat{\mathcal{O}}, \hat{\mathcal{E}} \rangle. \quad (53)$$

One may object that, strictly speaking, the finite differential operators  $\hat{\mathcal{O}}$  and  $\hat{\mathcal{E}}$  satisfy the symmetry equation (52) only when it is applied to formal eigenfunctions (physical or not) of the form  $\Psi = \Theta_E^M(\theta)\Phi_\epsilon(\phi)$ ,  $M = k\epsilon$ , that is, the set of functions where the operators  $\hat{X}^\pm$  are well defined (see Lissajous-1). However, simple arguments show that if a linear finite order partial differential operator in  $\theta$  and  $\phi$  annihilates the infinite linear space of all the formal eigenfunctions  $\Psi = \Theta_E^M(\theta)\Phi_\epsilon(\phi)$  then, this operator must identically be zero.

The algebraic structure of the polynomial symmetries can be obtained from that of the fundamental symmetries by plugging (46) and (47) into (48). The result is

$$\begin{cases} [\hat{H}_\phi, \hat{\mathcal{O}}] = 2n \hat{\mathcal{E}} + n^2 \hat{\mathcal{O}}, & [\hat{H}_\phi, \hat{\mathcal{E}}] = 2n \hat{\mathcal{O}} \hat{H}_\phi + n^2 \hat{\mathcal{E}} \\ [\hat{\mathcal{O}}, \hat{\mathcal{E}}] = -n \hat{\mathcal{O}}^2 \mp P_2(\hat{H}, \hat{H}_\phi) \end{cases} \quad (54)$$

where the signs  $-$  and  $+$  correspond to cases (a) and (b), respectively. As we know by construction, these symmetries are not independent. The constrain relation, obtained from (49), has the form

$$-\hat{\mathcal{O}}^2 \hat{H}_\phi - n \hat{\mathcal{O}} \hat{\mathcal{E}} + \hat{\mathcal{E}}^2 = \pm P_1(\hat{H}, \hat{H}_\phi). \quad (55)$$

Finally, we have arrived at a polynomial algebra of polynomial symmetries; the degree of this algebra is the same as in the classical case: for  $k = 1$  the algebra is quadratic, and for  $k \neq 1$  the degree is  $m + n - 1$ .

It is quite instructive to compare the formulas of the quantum symmetry algebras of this subsection with those of Subsection 2.2 corresponding to the classical algebras.

### 3.3 Hermitian properties of polynomial symmetries

Now, we will address the question of the Hermitian properties of the polynomial symmetries. As the Hermitian properties of  $\hat{X}^\pm$  in principle are not known, we can not make use of the definition given in (50) and (51) of  $\hat{\mathcal{O}}$  and  $\hat{\mathcal{E}}$  to find the Hermitian conjugate of the polynomial operators. Therefore, in order to find them we have to work with the following conditions:

- The Hamiltonian  $\hat{H}$  given in (34) is Hermitian with respect to the inner product

$$\langle \Psi_1, \Psi_2 \rangle = \int_{-\pi/2}^{\pi/2} d\phi \int_0^\pi d\theta \Psi_1(\phi, \theta)^* \Psi_2(\phi, \theta) \sin \theta. \quad (56)$$

So, the formal Hermitian conjugate of the differential operators are

$$(\partial_\theta)^\dagger = -(\partial_\theta + \cot \theta), \quad (\partial_\phi)^\dagger = -\partial_\phi. \quad (57)$$

- The Hermitian conjugate of a polynomial symmetry must be another polynomial symmetry. As  $\hat{O}$  is the minimum order nontrivial symmetry, we must have the following kind of Hermitian transformations:

$$\hat{O}^\dagger = (-1)^{m+n-1} \hat{O} + Q(\hat{H}, \hat{H}_\phi), \quad \hat{E}^\dagger = (-1)^{m+n} \hat{E} + \alpha \hat{O} + R(\hat{H}, \hat{H}_\phi) \quad (58)$$

where  $\alpha$  is a constant, and the polynomials  $Q(\hat{H}, \hat{H}_\phi)$  and  $R(\hat{H}, \hat{H}_\phi)$  must contribute in lower degrees less than precedent terms.

- The Hermitian properties of  $\hat{O}$  and  $\hat{E}$  must be consistent with the commutation rules (54).

We have found the following Hermitian rules that satisfy all the above requirements,

$$\hat{O}^\dagger = (-1)^{m+n+1} \hat{O}, \quad \hat{E}^\dagger = (-1)^{m+n} (\hat{E} + n \hat{O}). \quad (59)$$

In the following examples, we have checked, that indeed the polynomial symmetries fulfill the above Hermitian properties. We could change to an Hermitian/anti-Hermitian basis if we introduce the new polynomial symmetry

$$\hat{E}' = \hat{E} + \frac{n}{2} \hat{O}. \quad (60)$$

The commutation relation in the new basis are

$$\begin{cases} [\hat{H}_\phi, \hat{O}] = 2n \hat{E}', & [\hat{H}_\phi, \hat{E}'] = n(\hat{O} \hat{H}_\phi + \hat{H}_\phi \hat{O}) - \frac{1}{2} n^3 \hat{O} \\ [\hat{O}, \hat{E}'] = -n \hat{O}^2 \mp P_2(\hat{H}, \hat{H}_\phi) \end{cases} \quad (61)$$

and the restriction relation is

$$-\hat{O} \hat{H}_\phi \hat{O} + \hat{E}'^2 - \frac{n^2}{4} \hat{O}^2 = \pm (P_1(\hat{H}, \hat{H}_\phi) + \frac{n}{2} P_2(\hat{H}, \hat{H}_\phi)). \quad (62)$$

These new relations are explicitly consistent with the Hermitian rules

$$\hat{O}^\dagger = (-1)^{m+n-1} \hat{O}, \quad \hat{E}'^\dagger = (-1)^{m+n} \hat{E}'. \quad (63)$$

### 3.4 Examples of quantum polynomial symmetry algebras

Next we will show some examples for some values of  $k$ , just to see explicitly a few simple realizations of these algebras.

- Case  $m = 1, n = 1, k = 1$

$$\hat{\mathcal{O}} = -\cot \theta \cos \phi \partial_\phi - \sin \phi \partial_\theta$$

$$\hat{\mathcal{E}} = \cot \theta \sin \phi \hat{H}_\phi + \cos \phi \partial_\theta^2$$

$$P_1 = -\alpha^2 \hat{H} + (\alpha^2 - 1 + \hat{H}) \hat{H}_\phi - \hat{H}_\phi^2$$

$$P_2 = \alpha^2 + \hat{H} - 2 \hat{H}_\phi$$

- Case  $m = 1, n = 2, k = 1/2$

$$\hat{\mathcal{O}} = \frac{1}{2} \cot \theta \left[ \sin^2 \phi \hat{H}_\phi - \sin(2\phi) \partial_\phi + \cos^2 \phi \partial_\phi^2 \right] + \cos(2\phi) \partial_\theta + \sin(2\phi) \partial_{\theta\phi}^2$$

$$\begin{aligned} \hat{\mathcal{E}} = & -\frac{1}{2} \cot \theta \left[ \cos(2\phi) \hat{H}_\phi + \sin(2\phi) \partial_\phi \hat{H}_\phi \right] - \sin^2 \phi \partial_\theta \hat{H}_\phi + \sin(2\phi) \partial_{\theta\phi}^2 \\ & - \cos^2 \phi \partial_{\theta\phi\phi}^3 \end{aligned}$$

$$\begin{aligned} P_1 = & \alpha^2(\alpha^2 - 2) \hat{H} + \frac{1}{4} \left[ -4 + 10\alpha^2 - \alpha^4 + 4(5 - 2\alpha^2) \hat{H} \right] \hat{H}_\phi \\ & + \frac{1}{4} (-13 + 2\alpha^2 + 4\hat{H}) \hat{H}_\phi^2 - \frac{1}{4} \hat{H}_\phi^3 \end{aligned}$$

$$P_2 = \alpha^2 \left( 1 - \frac{\alpha^2}{2} \right) + 2(1 - 2\alpha^2) \hat{H} + (2\alpha^2 - 3) \hat{H}_\phi + 4\hat{H} \hat{H}_\phi - \frac{3}{2} \hat{H}_\phi^2$$

- Case  $m = 2, n = 1, k = 2$

$$\hat{\mathcal{O}} = -[3 + \cos(2\theta)] \csc^2 \theta \cos \phi \partial_\phi + 4 \cot^2 \theta \sin \phi \hat{H}_\phi - \cot \theta (\sin \phi \partial_\theta - 4 \cos \phi \partial_{\theta\phi}^2) + \sin \phi \partial_\theta^2$$

$$\hat{\mathcal{E}} = [3 + 2 \cos(2\theta)] \csc^2 \theta \sin \phi \hat{H}_\phi - 4 \cot^2 \theta \cos \phi \partial_\phi \hat{H}_\phi - \cot \theta (4 \sin \phi \partial_\theta \hat{H}_\phi - \cos \phi \partial_{\theta\phi}^2) - \cos \phi \partial_{\theta\theta\phi}^3$$

$$P_1 = \alpha^2 (2 - \hat{H}) \hat{H} + [4 - 20\alpha^2 + (8\alpha^2 - 10) \hat{H} + \hat{H}^2] \hat{H}_\phi + (52 - 16\alpha^2 - 8\hat{H}) \hat{H}_\phi^2 + 16 \hat{H}_\phi^3$$

$$P_2 = -4\alpha^2 + 2(4\alpha^2 - 1) \hat{H} + \hat{H}^2 + 8(3 - 4\alpha^2 - 2\hat{H}) \hat{H}_\phi + 48 \hat{H}_\phi^2$$

- Case  $m = 3, n = 1, k = 3$

$$\begin{aligned} \hat{\mathcal{O}} = & \frac{27}{2} \cot \theta \left[ (3 + \cos(2\theta)) \csc^2 \theta \sin \phi \hat{H}_\phi - 2 \cos \phi \cot^2 \theta \partial_\phi \hat{H}_\phi - 2 \cot \theta \sin \phi \partial_\theta \hat{H}_\phi \right] \\ & - \frac{3}{2} [15 \cos \theta + \cos(3\theta)] \csc^3 \theta \cos \phi \partial_\phi - [2 + \cos(2\theta)] \csc^2 \theta \sin \phi \partial_\theta \\ & + \cos \phi [18 \cot^2 \theta + 9 \csc^2 \theta] \partial_{\theta\phi}^2 + 3 \cot \theta \sin \phi \partial_\theta^2 - 9 \cos \phi \cot \theta \partial_{\theta\theta\phi}^3 - \sin \phi \partial_\theta^3 \end{aligned}$$

$$\begin{aligned} \hat{\mathcal{E}} = & 27 \cot^3 \theta \sin \phi \hat{H}_\phi^2 + \frac{3}{2} \csc^2 \theta \left[ (15 \cos \theta + \cos(3\theta)) \csc \theta \sin \phi \hat{H}_\phi \right. \\ & - 3 \left\{ 3(3 + \cos(2\theta)) \cot \theta \cos \phi \partial_\phi \hat{H}_\phi - 6 \cos^2 \theta \cos \phi \partial_{\theta\phi} \hat{H}_\phi \right. \\ & + \left. \sin \phi \left( 2(2 + \cos(2\theta)) \partial_\theta \hat{H}_\phi - \sin(2\theta) \partial_\theta^2 \hat{H}_\phi \right) \right\} \\ & \left. + \cos \phi [(2 + \cos(2\theta)) \csc^2 \theta \partial_{\theta\phi}^2 - 3 \cot \theta \partial_{\theta\theta\phi}^3 + \partial_{\theta\theta\theta\phi}^4] \right] \end{aligned}$$

$$\begin{aligned} P_1 = & \alpha^2 (-12 \hat{H} + 8 \hat{H}^2 - \hat{H}^3) + [-36 + 360\alpha^2 + (120 - 351\alpha^2) \hat{H} + (27\alpha^2 - 35) \hat{H}^2 \\ & + \hat{H}^3] \hat{H}_\phi + [-1737 + 2511\alpha^2 + (837 - 243\alpha^2) \hat{H} - 27 \hat{H}^2] \hat{H}_\phi^2 + (-4698 + 729\alpha^2 \\ & + 243 \hat{H}) \hat{H}_\phi^3 - 729 \hat{H}_\phi^4 \end{aligned}$$

$$\begin{aligned} P_2 = & 36\alpha^2 + (12 - 108\alpha^2) \hat{H} + (27\alpha^2 - 8) \hat{H}^2 + \hat{H}^3 + [-396 + 1377\alpha^2 \\ & + (459 - 486\alpha^2) \hat{H} - 54 \hat{H}^2] \hat{H}_\phi + [-3888 + 2187\alpha^2 + 729 \hat{H}] \hat{H}_\phi^2 - 2916 \hat{H}_\phi^3 \end{aligned}$$

## 4 Conclusions

This paper has been devoted to the algebraic structure of the ‘fundamental symmetries’ and their relation to polynomial symmetries for an example in the class of Lissajous systems. Explicit expressions of such symmetries as well as their algebraic structure have been obtained. In this process, we have applied the same method for both classical and quantum cases, showing the close similarities of the corresponding expressions and properties. We have chosen a very simple system to show clearly the main steps of our method, but it can be applied to a wide class of superintegrable systems. In this respect, work is in progress to prepare a systematic list of applications.

Next, we will briefly comment on the results and methods of some previous references dealing with symmetries of similar systems and the connection with our results. As it was mentioned in Lissajous–1, the Lissajous systems include the so called TTW system [7, 8] and a list of similar systems considered in Refs. [9, 10]. The classical superintegrability of such systems was proved in many references [11, 12, 10, 13, 14, 15, 16]. In general, the methods followed in these references were based on the solutions of the Hamilton-Jacobi equation or on action-angle variables arguments. The superintegrability of the quantum versions of many of these systems was studied by means of the recurrence properties of the solutions of the corresponding stationary Schrödinger equations and other techniques. These recurrence relations were applied to get polynomial symmetry operators [17, 18, 19].

As we have remarked in Lissajous–1 our method implement parallel procedures for classical and quantum systems, so it is essentially different from those of the above mentioned references. Our way to get symmetries is based on algebraic properties of the Hamiltonians as differential operators (or functions), in other words, on the existence of shift and ladder operators (or functions). Therefore, we have not used the explicit form of any solution to get symmetries but the other way round.

One of our main objectives has been to know in easy terms the origin of the algebraic relations for the classical and quantum polynomial symmetries. Besides this, in our approach one can directly appreciate how close are the expressions and properties of classical and quantum systems, see for instance (19) and (54). Finally, we have obtained the Hermitian relations of our polynomial symmetry algebras, which are consistent with the commutation relations.

Other references have considered the problem of superintegrability of classical systems on constant curvature spaces [21, 22] from a different point of view. Some other methods have been applied to display superintegrability properties, for instance in classical systems a kind of complex functions (similar to our shift and ladder functions) have been used in [23, 24]. A coalgebra procedure has been shown to be useful to find the symmetries of higher dimensional systems [20]. Other algebraic methods have also been developed in [25].

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